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CLASSES OF MODELS FOR SELECTED AXIOMATIC THEORIES OF CHOICE

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ABSTRACT

Adding a reversibility axiom to the other axioms of Luce's (1959) probabilistic ranking theory results in an impossibility theorem - that all alternatives in an alternative set are equally likely to be chosen (i.e., that preferences are random). This impossibility theorem is generally avoided by removing the reversibility axiom. Using simple algebraic methods such a modified theory is shown to contain a theorem similar to the impossibility result. These results are discussed within the framework of methematical model theory - model theory deals with the relations between sets of sentences (theories) and the structures which satisfy these sentences (models) - to illustrate the applicability of model theory as an analytic tool in theory development.

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Abstract

Adding a reversibility axiom to the other axioms of Luce's (1959) probabilistic ranking theory results in an impossibility theorem - that all alternatives in an alternative set are equally likely to be chosen (i.e. that preferences are random). This impossibility theorem is generally avoided by removing the reversibility axiom. Using simple algebraic methods such a modified theory is shown to contain a theorem similiar to the impossibility result. These results are discussed within the framework of mathematical model theory - model theory deals with the relations between sets of sentences (theories) and the structures which satisfy these sentences (models) - to illustrate the applicability of model theory as an analytic tool in theory development.

Considerable work by mathematical psychologists has been devoted to developing axiomatic theories of choices. In this paper, a particular set of probabilistic ranking theories (PRT) (Luce & Suppes, 1965) will be examined from a model-theoretic perspective. Probabilistic theories of choice are those which assume that an individual's choice responses are governed by probability mechanisms and ranking theories are those which attempt to explain (describe) relations between results of experiments in which individuals are asked to select one item from among a number of alternatives and experiments in which he is asked to rank order the alternatives.

Some of the most important theoretical work in the area of PRT may be found in Luce (1959). Unfortunately, nearly all attempts to develop PRT quickly lead to the apparently anomolous result to be shown below (Block & Marschak, 1960, p. 111; Luce, 1959, p. 69; Luce & Suppes, 1965, pp. 356-358).

First, however, the axioms of the theory must be carefully specified. In doing this, it is important (the reasons for this will become clear later) to distinguish between the "calculus axioms" and the "proper axioms. Calculus axioms are those containing no "extra-logical constants" (Braithwaite, 1959, p. 429) and may be thought of as providing the basic logic for manipulating the sentences in the theory. Proper axioms, on the other hand, are those containing non-vacuously extra logical constants and correspond to the "substantive" axioms for the theory.

The Axioms

The distinction between calculus and proper axioms will be made more clear by a specific consideration of the axiom sets used by Luce (1959). The calculus axioms for Luce's PRT are those of the probability calculus and are set out as below:

ca(1) For $S \subset T$, $0 \leq P_T(S) \leq 1$

ca(2) $P_T(T) = 1$

ca(3) If $R, S \subset T$ and $R \cap S = \emptyset$, then $P_T(R \cup S) = P_T(R) + P_T(S)$.

In axiom ca(1), T is a finite subset of the universal set U, and it is asserted that for any subset S of T, the probability that some element x (of T) is in S (designated by $P_T(S)$) is greater than or equal to zero and less than or equal to one. Axiom ca(2) tells us the probability that some element x (of T) is in T equals one. Axiom ca(3) says that if R and S are two subsets of T such that the intersection of R and S is the empty set ($\emptyset$), the probability that some element x (of T) is in R union S equals the sum of the probabilities that x is in R or S.

The next thing to be done is to specify the proper axioms. Let us restrict ourselves to situations in which an individual is faced with three alternatives, and let $A = \{a, b, c\}$ denote an arbitrary set with three elements. Then $P_A(x)$ will be the probability that an individual will choose x as his most preferred alternative from the set A (that is, the variable x may take on the value a, b, or c). A might be the set of dishes on a menu with "a" representing roast beef, "b" steak, and "c" hamburger. $P_A(a)$ would then be the probability of preferring roast beef out of the alternatives in A.

Further, let $p(x, y)$ denote the probability that an individual prefers alternative x to alternative y in the reduced set $\{x, y\}$. Finally, $\bar{p}(x, y, z)$ will be the probability of ranking the three alternatives from most preferred to least preferred in the order x, y, z. The notation $p(x, y)$ and $\bar{p}(x, y, z)$ is an abbreviation of the set notation introduced above. For example, $p(x, y)$ replaces $P_{\{x, y\}}(x)$, and statements like $P_{\{x, y\}}(x) + P_{\{x, y\}}(y) = P_{\{x, y\}}(\{x, y\}) = 1$ are rewritten as $p(x, y) + p(y, x) = 1$. It will be assumed throughout this paper that none of the functions P_A, p, $\bar{p}$ takes the value zero for any argument.

With this notation several proper axioms relating how alternatives are ranked can be written. The first such axiom will be the <u>decomposition</u> axiom:

pa(1) (Decomposition): $\bar{p}(x, y, z) = P_A(x)p(y, z)$

The decomposition axiom simply states that the probability of preferring x to y and y to z may be decomposed into the product of the probability of choosing x as the most preferred alternative in A and the probability of preferring y to z.

The second assumption is the consistency axiom:

pa(2) (Consistency): (i) $P_A(x) = \bar{p}(x, y, z) + \bar{p}(x, z, y)$

(ii) $p(x, y) = \bar{p}(x, y, z) + \bar{p}(x, z, y) + \bar{p}(z, x, y)$

The intended interpretation of this axiom should be fairly self evident. It simply asserts that there is consistency between choice rankings and that adding irrelevant alternatives will not change the ranking. An important concern then might be in alternative ways of calculating $p(x, y)$. That is, what is the probability of say, preferring steak to roast beef when hamburger is not available? Proper axioms (1) and (2) together with the calculus axioms can be used to prove:

Theorem 1: $p(x, y) = \dfrac{P_A(x)}{P_A(x) + P_A(y)}$

Proof:

(1) $p(x, y) = P_A(x)p(y, z) + P_A(x)p(z, y) + P_A(z)p(x, y)$

by pa(2)(ii) and pa(1)

(2) $p(x, y)\ (1 - P_A(z)) = P_A(x)(p(y, z) + p(z, y)) = P_A(x)$

since $p(y, z) + p(z, y) = 1$

(3) $p(x, y) = \frac{P_A(x)}{(1 - P_A(z))}$ $1 - P_A(z)$ is assumed to be $\neq 0$

(4) $1 - P_A(z) = P_A(x) + P_A(y)$ by ca(2) and ca(3)

(5) $p(x, y) = \frac{P_A(x)}{P_A(x) + P_A(y)}$ by substitution of (4) into (3).

Theorem 1, in a more general form, is known as Luce's choice axiom. It can easily be shown that Theorem 1 together with proper axiom 1 or proper axiom 2 implies the other proper axiom. This can be shown to be true where the cardinality of the alternative set A is any finite n and not only for n equal to three. Equally important, there is empirical evidence suggesting that Theorem 1 provides highly accurate predictions of reduced set choice probabilities for certain types of alternative sets.

Now, let $P_A*(x)$ be the probability of choosing x as the least preferred alternative in A. Similarly, let $p*(x, y) = P*_{\{x, y\}}(x)$ and let $\overline{p}*(x, y, z)$ be the probability of having the rank order x, y, z when asked to rank from least preferred to most preferred. The last proper axiom, the reversibility axiom, can now be stated:

pa(3) (Reversibility): (i) $\overline{p}*(x, y, z) = \overline{p}(z, y, x)$

(ii) $p*(x, y) = p(y, x)$

(iii) pa(1) and pa(2) hold for P_A^*, p^*, $\overline{p}^*$

The reversibility axiom simply states that the probability of getting a certain ranking when going from most preferred to least preferred is the same as getting the reverse of that ranking when the criterion is going from least to most preferred.

Surprisingly, it is possible to prove that if decomposition, consistency, and reversibility hold, all alternatives are preferred with equal probability, that is, that preferences are random. In other words:

Theorem 2: $P_A(a) = P_A(b) = P_A(c) = P_A^*(a) = P_A^*(b) = P_A^*(c) = 1/3$.

Proof: It will be sufficient to show $P_A(x) = P_A(y)$, since axiom pa(3) implies the same result for P_A^*.

(1) $\bar{p}(x, y, z) = P_A(x)p(y, z)$ — by pa(1)

(2) $\bar{p}^*(z, y, x) = P_A^*(z)p^*(y, x) = P_A^*(z)p(x, y)$ — by pa(3)(iii) and (ii)

(3) $P_A(x)p(y, z) = P_A^*(z)p(x, y)$ — by pa(3)(i) applied to steps (1) and (2)

(4) $P_A(x) = \dfrac{P_A^*(z)p(x, y)}{p(y, z)}$ — by rewriting (3)

(5) $P_A(y) = \dfrac{P_A^*(z)p(y, x)}{p(x, z)}$ — by interchanging x and y in (4)

(6) $p(x, y) = \dfrac{P_A(x)}{P_A(x) + P_A(y)}$ — Theorem 1

(7) $p(x, y) = \dfrac{\dfrac{P_A^*(z)p(x, y)}{p(y, z)}}{\dfrac{P_A^*(z)p(x, y)}{p(y, z)} + \dfrac{P_A^*(z)p(y, x)}{p(x, z)}}$ — by substituting (4) and (5) into (6)

(8) $p(x, z) = p(y, z)$ — by simplification of (7)

(9) $p(x, y) = P(z, y)$

$p(y, x) = p(z, x)$ — by interchanging the positions of x, y, z in (8)

(10) $p(z, x) = p(z, y)$ — since $p(z, x) = 1 - p(x, z) = 1 - p(y, z) = p(z, y)$

(11) $p(x, y) = p(y, x) = 1/2$ — by (9) and (10)

(12) $P_A(x) = P_A(y)$ — substituting 1/2 for $p(x, y)$ in (6). ∎

(For a proof of Theorem 2 for the n-alternative case, see Luce & Suppes, 1965, pp. 356-358.)

These results seem contrary to experience. Unfortunately, the culprit is not obvious. Proper axioms (1) and (2) are the most likely suspects. However, together they imply Luce's choice axiom (Theorem 1) which has considerable empirical support. Moreover, Luce's choice axiom together with either proper axiom 1 or 2 implies the other. Thus if it is wished to retain Theorem 1 as a theorem, both proper axiom (1) and (2) must be retained. If, on the other hand, Theorem 1 is made an axiom, then both pa(1) and (2) must be thrown out. Axiom (3) (reversibility) seems to be some psychologist's favorite candidate for elimination. Their argument is that it makes no operational sense to ask a person to pick his "least preferred" alternative from some set of (homogeneous) alternatives. At least at first glance this claim appears very unconvincing (though the reader who sees potential merit in it is referred to Luce & Suppes, 1965, p. 358, where the position is spelled out). One need only introspect for a moment on the alternative set consisting of a thousand dollar bill, a hundred dollar bill, and a one dollar bill. Few people would find it difficult to pick out their least preferred alternative.

Theorem 2 is not, it would seem, a trivial result. It was obtained by making three apparently innocuous assumptions about how individuals related choice probabilities over a three alternative set. Yet these proper axioms together imply that unless $P_A(a) = P_A(b) = P_A(c)$, any individual who ranks the alternative elements of a three element set in the same fashion regardless of whether he ranks them from most preferred to least preferred or from least preferred to most preferred is exhibiting behavior which is inconsistent with that described by proper axioms (1) and (2).

What then is the theorist to do in the face of such an anomaly? One option is, of course, to ignore proper axiom (3) and simply develop the theory using proper axioms (1) and (2). A second option is to investigate the mathematical structure underlying the axioms in question to see whether results which appear disturbing result directly from equally disturbing (but more subtle) properties of the class of models satisfying these axioms (and therefore theorems as well). Indeed, this appears to be the approach suggested by Luce and Suppes (1965) when they wrote:

"They (criticisms of probabilistic choice theories) suggest that we cannot hope to be completely successful in dealing with preferences until we include some mathematical structure over the set of outcomes that, for example, permits us to characterize those outcomes that are simply substituable for one another and those that are special cases of others. Such functional and logical relations among the outcomes seem to have a sharp control over the preference probabilities, and they cannot long be ignored [p. 337]." While it is not completely clear what is meant by the above passage, it does seem they are suggesting a closer investigation of the mathematical (logical) structures underlying various theories of choice. This paper represents an attempt to explore this suggestion for probabilistic ranking theories which contain proper axioms (1) and (2).

Model Theory

In most of the behavioral science literature, no clear distinction is drawn and maintained between models and theories. Indeed, perhaps the most common practice is to use "model" and "theory" interchangeably as synonyms as in Tversky (1972): "Since the present theory describes choice as an elimination process governed by successive selection of aspects, it is called the elimination-by-aspects (EBA) model [p. 285]". It would seem that, from Professor Tversky's perspective, it could as well have been called the EBA theory.

There certainly is nothing wrong with having synonyms for such frequently used words as "theory." However, when "theory" is used in its technical sense, there is a clear distinction which can be made between "models" and "theories," and this distinction has useful consequences for the topic at hand. A theory, in its technical sense, is a set of sentences which is closed under deduction; that is, the set contains any sentence that is logically implied by any other sentences in the set. This concept requires some preassigned logical framework (e.g., first-order predicate calculus) (Quine, 1968, p. 281). Whenever an axiom system is proposed (as in Tversky, 1972), this usage of "theory" is implied. On the other hand, a nontechnical theory is simply a set of sentences asserted to be true. For example, the entire body of knowledge on some subject may be referred to as the theory of that subject, as in the phrase "choice theory."

A corresponding technical notion of a model for a set of sentences (theory) is a mathematical structure which satisfies those sentences. Thus a model is a set-theoretic structure while a theory is a collection of sentences in some language. More specifically, a set-theoretic structure M is a set of elements (objects), $A = \{a_1, a_2, \ldots\}$, together with a set of relations of order i, $P_1^{i_1}$, $P_2^{i_2}$, ..., and may be expressed

$M = \langle A; P_1^{i_1}, P_2^{i_2}, \ldots, P_n^{i_n}, \ldots \rangle$.

A formal language L in which properties of M can be expressed will consist of formulas generated by a specified set of rules, say the predicate calculus, from an alphabet consisting of relation symbols (R_1, R_2,..), variable symbols (x_1, x_2,...), connectives ($\neg, \vee, \wedge, \ldots$), and quantifiers ($\forall, \exists$). Since functions and constants are special kinds of relations, function symbols (f_1, f_2,...) and constant symbols (c_1, c_2,...) will also be used in L. The language L will be assumed to be first order, that is, its variables range over the elements of A (as opposed to ranging over the subsets of A, or sets of subsets, etc.). Sentences in L are formulas containing no free variables.

Let T be a set of axioms in a language L. If φ is a mapping of constant symbols occurring in T into the set of objects A, and also a mapping of relation symbols occurring in T into the set of relations in M, then M provides an interpretation of T under φ. If this interpretation results in the sentences in T being true, then M is said to satisfy T and M is a model of the axiom set T. A model for a set of axioms then, is a set-theoretic mathematical structure which interprets the axioms in such a way that the axioms are true.

The distinction just made between objects and symbols denoting objects (constants) and between relations and relation symbols should be emphasized. The reason for this distinction is that each mapping onto the objects and relations in a structure M provides an interpretation of the symbols in T. This is important since (as will be shown) a given axiom set can have more than one interesting interpretation, and only some of them will be models of the set.

One of the most obvious problems with the above definition of model is what is meant by a sentence being "true." Rather than provide an extended discussion of truth, the reader is referred to Tarski (1944). The important question here is not how do we know whether a particular sentence is in fact true but rather what is meant by asserting a sentence to be true. This latter semantic question is treated in considerable detail by Tarski for important classes of formal languages (including those to be dealt with in this paper).

In order to make this definition of model more clear, consider a very simple theory T' which contains only two proper axioms:

A1: $(\forall x_1)\ \neg(x_1 R x_1)$

A2: $(\forall x_1)(\forall x_2)(\forall x_3)[(x_1 R x_2 \wedge x_2 R x_3) \supset x_1 R x_3]$.

Consider further the following two mathematical structures:

M^*: $\langle A; P^2 \rangle$ where A is a finite set of alternatives and P^2 is the binary relation "is preferred to"

and

M^{**}: $\langle L; F^2 \rangle$ where L is the set of living males and F^2 is the binary relation "is the father of".

If the symbol R is mapped onto P^2, and the variables are assumed to range over A, then A1 would read as "for all alternatives in the set A, it is never the case that an alternative in A is preferred to itself." Axiom A2 would read: "For any triple of alternatives in the set A, if the first alternative is preferred to the second, and the second is preferred to the third, then the first alternative is preferred to the third." To claim M^* to be a model of T' is to assert the truth of these two sentences (A1 and A2). Further, Tarski (1944) shows that asserting a sentence to be true is equivalent to saying it is satisfied by all its objects. Again, there exists no algorithm for determining whether a particular sentence is in fact satisfied by all its objects. However, to assert that T' is modeled by M^* is to say that each sentence in T' is satisfied by all its objects.

Let us now examine the relation between the structure M^{**} and the sentences in T'. Do we want to assert that M^{**} is a model of T'? In this case the φ function maps the relation symbol R onto the relation F^2. Interpreting A1 with M^{**} results in the sentence:

"For all males in the set of all living males, it is never the case that a male is the father of himself."

To assert that M^{**} is a model for T' is to assert this to be a true sentence. And, indeed, the sentence is empirically true. However, we must be careful not to move hastily from this observation to asserting that M^{**} is a model for T'. The definition of a model requires that <u>all</u> the axioms be true when interpreted by a model. Consider A2. Under M^{**} we have the following sentence:

"For any three males in L, if $male_1$ is the father of $male_2$, and $male_2$ is the father of $male_3$, then $male_1$ is the father of $male_3$."

Again, to assert M^{**} is a model for T' is to assert the truth of this sentence. Yet this sentence is empirically untrue. Indeed, an ordinary language translation of this sentence would result in the assertion that a grandfather is the father of his grandson. The reason "is preferred to" seems a satisfactory interpretation of R and "is the father of" does not is that "is preferred to" is generally thought to be a transitive relation (as asserted by A2) and "is the father of" is not transitive. Thus the structure M^{**} is not a model for T'.

Another transitive relation is "is greater than." If the letter "I" denotes the set of integers, and ">" denotes "is greater than," then the structure $\langle I, > \rangle$ is a model for T'. A third transitive relation "is greater than or equal to" may be denoted by "$\geq$". Consider whether the structure $\langle I, \geq \rangle$ is a model of T'. Clearly axiom A2 is true with this interpretation; however, A1 reads as follows:

"For any integer, it is never the case that the integer is greater than or equal to itself."

Most of us would assert this sentence to be false and not allow $\langle I, \geq \rangle$ as a model for T'.

Hopefully, these overly simplistic examples provide a general sense of how the terms "model" and "theory" are being used in this paper. Moreover, it should be clear from the above discussion that it is possible to develop a theory of models. In Robinson's (1963) words: "Model theory deals with the relations between the properties of sentences or sets of sentences specified in a formal language on one hand, and of the mathematical structures or sets of structures which satisfy these sentences, on the other hand [p. 1]".

Note the similarity between Robinson's definition of model theory and Luce and Suppes' quote in the previous section. In the next section a result analogous to Theorem 2 (proved using proper axioms (1) - (3)) will be shown to exist for the more commonly encountered proper axioms (1) and (2), and some of the model theoretic concepts introduced here will be used to analyze these two axiom sets.

Algebraic Results

This section begins with some algebraic manipulations on the equations in proper axioms (1) and (2). The results, theorems 3 and 4, together with theorems 1 and 2, will then be discussed in model-theoretic terms.

In the three alternative case there are six possible rankings of those alternatives. For notational convenience the corresponding probability values will be denoted by the set of symbols $\mathcal{T} = \{a_1, a_2, b_1, b_2, c_1, c_2\}$ as follows:

$\bar{p}(x, y, z) = a_1$	$\bar{p}(y, x, z) = b_2$
$\bar{p}(x, z, y) = a_2$	$\bar{p}(z, y, x) = c_1$
$\bar{p}(y, z, x) = b_1$	$\bar{p}(z, x, y) = c_2$

The following derivation in terms of the value $a_1 = \bar{p}(x, y, z)$ may be applied to each of the elements in $\mathcal{T}$. In the decomposition formula for $\bar{p}(x, y, z)$,

$$\bar{p}(x, y, z) = F_A(x) \cdot p(y, z). \tag{1}$$

(1) may be expanded by making substitutions on the right according to the formulas in axiom pa(2), yielding

$$\bar{p}(x, y, z) = (\bar{p}(x, y, z) + \bar{p}(x, z, y))(\bar{p}(x, y, z) + \bar{p}(y, z, x) + \bar{p}(y, x, z)). \tag{2}$$

Substituting the notation from above yields the equation

$$a_1 = (a_1 + a_2)(a_1 + b_1 + b_2). \tag{3}$$

Expanding the right side of (3)

$$a_1 = a_1^2 + a_1a_2 + a_1b_1 + a_1b_2 + a_2b_1 + a_2b_2, \tag{4}$$

and collecting terms

$$0 = a_1^2 + (a_2 + b_1 + b_2 - 1)a_1 + a_2(b_1 + b_2), \tag{5}$$

yields a quadratic expression in a_1. Applying the quadratic formula to solve for a_1 results in

$$a_1 = \frac{1 - (a_2 + b_1 + b_2) \pm \sqrt{(1 - (a_2 + b_1 + b_2))^2 - 4a_2(b_1 + b_2)}}{2}. \tag{6}$$

And letting $g(y_1, y_2, y_3)$ denote the function represented in (6),

$$g(y_1, y_2, y_3) = \frac{1 - (y_1 + y_2 + y_3) \pm \sqrt{(1 - (y_1 + y_2 + y_3))^2 - 4y_1(y_2 + y_3)}}{2}, \tag{7}$$

the results of applying the above procedure (1 - 6) to each of the six rankings are abbreviated as follows:

$$\begin{aligned} a_1 &= g(a_2, b_1, b_2) & b_2 &= g(b_1, a_1, a_2) \\ a_2 &= g(a_1, c_1, c_2) & c_1 &= g(c_2, b_1, b_2) \\ b_1 &= g(b_2, c_1, c_2) & c_2 &= g(c_1, a_1, a_2). \end{aligned} \tag{8}$$

Making the natural assumption that all the probability functions are real-valued, the quantity under the radical in (7) must be nonnegative, thus yielding six inequalities of the form

$$(1 - (y_1 + y_2 + y_3))^2 - 4y_1(y_2 + y_3) \geq 0. \qquad (9)$$

In the next section, the consequences of varying this assumption are considered.

Suppose now that the values in $\mathcal{T}$ are not all equal. What are the implications of the equations and inequalities derived above upon the range of values that elements of $\mathcal{T}$ may assume? Theorems 3 and 4 and Inequality (17) show that even after removing axiom pa(3), the elements of $\mathcal{T}$ can take on relatively few values.

In order to further simplify the algebraic calculations, let the variable symbol α be defined in the following equation:

$$y_1 + y_2 + y_3 = 1/2 + \alpha . \qquad (10)$$

Substituting (10) into (9) gives

$$(1 - (1/2 + \alpha))^2 - 4y_1(y_2 + y_3) \geq 0, \qquad (11)$$

and rewriting (10) as $(y_2 + y_3) = 1/2 + \alpha - y_1$ gives

$$(1 - (1/2 + \alpha))^2 - 4y_1(1/2 + \alpha - y_1) \geq 0 \text{ ,and} \qquad (12)$$

$$y_1^2 - (1/2 + \alpha)y_1 + \frac{(1/2 - \alpha)^2}{4} \geq 0 \quad . \qquad (13)$$

Applying the quadratic formula again, this time to find the zeros in (13), the expression on the left in (13) is shown to take on the value zero when

$$y_1 = (1/4 + \alpha/2) \pm \sqrt{\alpha/2} \qquad (0 \leq \alpha < 1/2), \qquad (14)$$

and hence (13) is satisfied when the value of y_1 is outside the interval

$$\left(\frac{1}{4} + \frac{\alpha}{2} - \sqrt{\frac{\alpha}{2}}, \frac{1}{4} + \frac{\alpha}{2} + \sqrt{\frac{\alpha}{2}}\right).$$

Since only six different sets of values for the arguments of the g function are of interest (cf. (8)), the corresponding six values of α can be denoted as follows:

$$\begin{array}{ll} a_2 + b_1 + b_2 = 1/2 + \alpha_{a_2} & b_1 + a_1 + a_2 = 1/2 + \alpha_{b_1} \\ a_1 + c_1 + c_2 = 1/2 + \alpha_{a_1} & c_2 + b_1 + b_2 = 1/2 + \alpha_{c_2} \\ b_2 + c_1 + c_2 = 1/2 + \alpha_{b_2} & c_1 + a_1 + a_2 = 1/2 + \alpha_{c_1} . \end{array} \qquad (15)$$

Observe that $\alpha_{a_1} = -\alpha_{a_2}$, $\alpha_{b_1} = -\alpha_{b_2}$, $\alpha_{c_1} = -\alpha_{c_2}$ and that all the α's are in the interval $(-1/2\ \ 1/2)$.

Theorem 3: If $\alpha_{a_1} = \alpha_{a_2} = \ldots = \alpha_{c_2} = 0$, then all the values $a_1, a_2, \ldots, c_2$ are equal.

Proof: From (15), $a_1 = b_2$, $a_2 = c_2$, and $b_1 = c_1$.

Substituting a_1 for b_2 in (3), yields

(i) $a_1 = (a_1 + a_2)(2a_1 + b_1)$.

Using the same procedure as above (1 - 6), an inequality analogous to (9) can be derived from (i):

(ii) $(1 - (2a_2 + b_1))^2 - 8a_2b_1 \geq 0,$

which can be rearranged as

(iii) $a_2^{\,2} - (b_1 + 1)a_2 + \left(\frac{b_1 - 1}{2}\right)^2 \geq 0.$

Applying the quadratic formula again results in

(iv) $a_2 \leq 1/2 + \frac{b_1}{2} - \sqrt{b_1}.$

This inequality may also be derived for the pairs $\langle b_1, b_2\rangle$ and $\langle b_2, a_2\rangle$, that is

(v) $b_1 \leq 1/2 + \frac{b_2}{2} - \sqrt{b_2}$ and

(vi) $b_2 \leq 1/2 + \frac{a_2}{2} - \sqrt{a_2}.$

The inequalities (iv) - (vi) together with $a_2 + b_1 + b_2 = 1/2$ imply that each of the values a_2, b_1, b_2 is less than .19. For example, if $a_2 = .19$, then (vi) implies $b_2 < .16$, and hence $b_1 = 1/2 - (a_2 + b_2) > .15$. However, $b_1 > .15$ implies $a_2 < .19$ since the function $(1/2 + b_1/2 - \sqrt{b_1})$ is a strictly decreasing function in the interval [0, 1] and its value at $b_1 = .15$ is less than .19. This kind of contradiction can be derived if any of the values in $\mathcal{T}$ is $\geq .19$. Hence all six values must fall in the interval (.12, .19).

Now it is straightforward to show that (i) implies $a_1 = a_2$. Let $a_2 = a_1 + \varepsilon$ where $0 < |\varepsilon| < .07$. Then

$$
\begin{aligned}
a_1 &= (a_1 + a_2)(2a_1 + b_1) \\
&= (2a_1 + \varepsilon)(1/2 - \varepsilon) \qquad \text{since } b_1 + a_1 + a_2 = 1/2 \\
&= a_1 + \varepsilon\,(1/2 - \varepsilon - 2a_1)
\end{aligned}
$$

which implies $(1/2 - \varepsilon - 2a_1) = 0$, and hence $a_1 = 1/4 - \varepsilon/2 > .215$ Therefore $\varepsilon = 0$ and $a_1 = a_2$. The other equalities are derived similarly. ∎

Thus the assumption that not all values in $\mathcal{T}$ are equal implies that at at least one of the α's is greater than zero. For definiteness assume that $\alpha_{a_1} > 0$, and let α denote this particular value. Substituting a_1 for y_1 in (13) yields

$$a_1^2 - (1/2 + \alpha)a_1 + \frac{(1/2 - \alpha)^2}{4} \geq 0 \,, \tag{16}$$

which implies (cf. 14)

$$a_1 \leq (1/4 + \alpha/2) - \sqrt{\frac{\alpha}{2}}, \text{ or } a_1 \geq (1/4 + \frac{\alpha}{2}) + \sqrt{\frac{\alpha}{2}} \,. \tag{17}$$

That is, given the value of $\alpha_{a_1} > 0$, the value of a_1 lies outside an interval of length $\sqrt{2\alpha_{a_1}}$ centered at $(1/4 + \frac{\alpha_{a_1}}{2})$. By the symmetry of the expression in (9), the sum $(c_1 + c_2)$ must also lie outside that interval. For example, if $\alpha_{a_1} = 1/8$, then either $a_1 \leq 1/16$ and $c_1 + c_2 \geq 9/16$ or $a_1 \geq 9/16$ and $c_1 + c_2 \leq 1/16$.

Does this result make a difference? The following detailed example illustrates the implications of (17). It is only loosely analogous to a choice experiment in Luce's sense; however, intuition suggests that axioms pa(1) and pa(2) should hold.

Imagine three barrels, labelled U, V, W, each of which contains ten balls of varying size. Barrel U contains 5 balls of size u_1, 3 balls of size u_2, and 2 balls of size u_3. V contains 4 balls of size v_1, 3 of size v_2, and 3 of size v_3. W contains 1 ball of size w_1, 3 of size w_2, and 6 of size w_3. The relative sizes of the u, v, and w's are

$u_1 > v_1 > w_1 > u_2 > v_2 > w_2 > u_3 > v_3 > w_3$.

An event will consist of drawing three balls, one from each barrel. There are a number of probability functions one could compute in this situation using elementary notions of probability theory (i.e., expressions implied by ca 1-3). Let $q_1(X)$ denote the probability that the ball drawn from the X barrel (X = U, V, or W) is the largest of the three drawn in a given event. Let $q_2(X, Y)$ be the probability that the ball drawn from X is bigger than the ball drawn from Y. And finally, let $q_3(X, Y, Z)$ denote the probability that the sizes of the three balls are ordered $x > y > z$.

The values of the three functions for this example are as follows:

$q_1(U) = .698$	$q_2(U, V) = .740$	$q_3(U, V, W) = .596$
$q_1(V) = .254$	$q_2(U, W) = .890$	$q_3(U, W, V) = .102$
$q_1(W) = .048$	$q_2(V, W) = .850$	$q_3(V, W, U) = .062$
	$q_2(V, U) = .260$	$q_3(V, U, W) = .192$
	$q_2(W, U) = .110$	$q_3(W, V, U) = .006$
	$q_2(W, V) = .150$	$q_3(W, U, V) = .042$

Using the letters $a_1, a_2, \ldots, c_2$ as before, we have $a_1 = .596$, $a_2 = .102$, $b_1 = .062$, and $b_1 + a_1 + a_2 = .76 = 1/2 + .26 = 1/2 + \alpha_{b_1}$. Applying (17) to b_1,

$$b_1 \leq \frac{1}{4} + \frac{.26}{2} - \sqrt{\frac{.26}{2}} \leq .38 - .36 = .02$$

or

$$b_1 \geq \frac{1}{4} + \frac{.26}{2} + \sqrt{\frac{.26}{2}} \geq .38 + .36 = .74$$

However, b_1 does not satisfy either inequality, implying that the functions q_1, q_2, and q_3 do not satisfy axioms pa(1) and pa(2).

For the case being discussed in which at least one of the α's is assumed greater than zero, an even stronger result than (17) can be shown.

Theorem 4: Let $\alpha_{a_1} > 0$. Then given two values a_1 and α_{a_1} (satisfying 17), there exist exactly two combinations of values for a_2, b_1, b_2, c_1, c_2 such that pa(1) and pa(2) are satisfied.

Proof: To show that a_2 depends only upon a_1 and α_{a_1}, combine (7) and (8) to get

$$\text{(i)}\quad a_2 = \frac{1 - (a_1 + c_1 + c_2) \pm \sqrt{(1 - (a_1 + c_1 + c_2))^2 - 4a_1(c_1 + c_2)}}{2} .$$

From (15),

$$\text{(ii)}\quad c_1 + c_2 = 1/2 + \alpha_{a_1} - a_1 .$$

Substituting (ii) into (i) gives

$$\text{(iii)}\quad a_2 = \frac{(1/2 - \alpha_{a_1}) \pm \sqrt{(1/2 - \alpha_{a_1})^2 - 4a_1(1/2 + \alpha_{a_1} - a_1)}}{2} .$$

To derive a formula for c_2, begin again as in (i)

$$\text{(iv)}\quad c_2 = \frac{1 - (c_1 + a_1 + a_2) \pm \sqrt{(1 - (c_1 + a_1 + a_2) - 4c_1(a_1 + a_2)}}{2} ,$$

and substitute (ii) into (iv). The resulting expression can be simplified to

$$\text{(v)}\quad (c_2 + (a_2 + \alpha_{a_1} - 1/2))^2 = (c_2 - (a_2 + \alpha_{a_1} - 1/2))^2 + 4(c_2 + a_1 - \alpha_{a_1})(a_1 + a_2).$$

The squared terms cancel in (v), which can then be solved for c_2:

$$\text{(vi)}\quad c_2 = \frac{(1/2 + \alpha_{a_1} - a_1)(a_1 + a_2)}{(1/2 - \alpha_{a_1} + a_1)} .$$

Solving (ii) for c_1 gives

$$\text{(vii)}\quad c_1 = 1/2 + \alpha_{a_1} - a_1 - c_2 .$$

Formulas for b_2 and b_1 are derived in a similar way, using another equation from (15)

$$\text{(viii)}\quad b_1 + b_2 = 1/2 - \alpha_{a_1} - a_2 :$$

$$\text{(ix)}\quad b_2 = \frac{(1/2 - \alpha_{a_1} - a_2)(a_1 + a_2)}{(1/2 + \alpha_{a_1} + a_2)} .$$

Observe that two values are generated for a_2, but that for each of these values there is a unique set of values for b_1, b_2, c_1, and c_2. ∎

As an example of a set of values obtainable according to the formulas in theorem 4, let $a_1 = 7/64$ and $\alpha a_1 = 1/32$. Then

$$a_2 = \frac{9}{64} \quad b_1 = \frac{21\cdot 27}{43\cdot 64} \quad b_2 = \frac{21}{43\cdot 4} \quad c_1 = \frac{21\cdot 27}{37\cdot 64} \text{ and } c_2 = \frac{27}{37\cdot 4},$$

or

$$a_2 = \frac{21}{64} \quad b_1 = \frac{9\cdot 27}{55\cdot 64} \quad b_2 = \frac{9\cdot 28}{55\cdot 64} \quad c_1 = \frac{27\cdot 9}{37\cdot 64} \text{ and } c_2 = \frac{27\cdot 28}{37\cdot 64}.$$

Consider now a model-theoretic framework in which the theorems proven above in the axiomatic choice theory can be discussed. The first step is to define a first-order language L adequate to express the axioms and theorems. L will contain the following components:

(a) Relation symbols to represent $\geq$, $\mathbb{R}$

(b) Function symbols with the appropriate number of arguments for the probability functions P_A, p, $\bar{p}$, P_A^*, p^*, $\bar{p}^*$

(c) Function symbols for $+$, $-$, $\cdot$, $\div$, $\sqrt{}$

(d) Constant symbols for 0, 1

(e) Variables $x_1, x_2, \ldots$

(f) Logical connectives $\neg, \vee, \wedge, \supset, =$

(g) Quantifiers $\forall, \exists$.

$\mathbb{R}$ is the set of real numbers and "=" is treated as a logical connective meaning identity. A relation symbol for $\mathbb{R}$ is needed so that the sets A and $\mathbb{R}$ can be distinguished, that is, so that variables can be quantified over just A or just $\mathbb{R}$. Generally the logical symbols remain implicit and L may be described by writing a vector of non-logical symbols, analogous to the notation for structures. (Function symbols will have a superscript denoting the number of arguments f^n, even though f^n represents an $n + 1$ place relation.) The language L is given by

$$L = \langle R_1^2, R_2^1; f_1^1, f_2^2, f_3^3, f_4^1, f_5^2, f_6^3, f_7^2, f_8^2, f_9^2, f_{10}^2, f_{11}^1; c_1, c_2\rangle$$

where the symbols are written in order corresponding to the order of their intended interpretations in (a) - (d) above. That is, R_1^2 corresponds to "$\geq$", f_4^1 to "P_A^*", f_7^2 to "+", etc.

As an example of a formal sentence in L, the decomposition axiom pa(1) can be translated

$$\text{pa}(1):\quad (\forall x_1)(\forall x_2)(\forall x_3)[(\neg x_1 = x_2 \wedge \neg x_1 = x_3 \wedge \neg x_2 = x_3 \wedge \neg R_2^1(x_1) \wedge \neg R_2^1(x_2) \wedge \neg R_2^1(x_3)) \supset f_3^3(x_1, x_2, x_3) = f_9^2(f_1^1(x_1), f_2^2(x_2, x_3))].$$

In addition to pa(1) - (2) there are a number of implicit and explicit assumptions employed in the first part of this section which should be stated as proper axioms. Implicit are all the field axioms for real numbers such as associativity, distributivity, existence of 0 and 1, and the basic axioms for an order relation. These will be referred to as the axioms for an ordered field. Second, it was assumed that the set A contained exactly three elements, and further that the square root function was defined only on nonnegative numbers. These axioms may be written as follows:

pa(4) Axioms for ordered field

$$\text{pa}(5)\quad (\exists x_1)(\exists x_2)(\exists x_3)[(\neg x_1 = x_2 \wedge \neg x_1 = x_3 \wedge \neg x_2 = x_3 \wedge \neg R_2^1(x_1) \wedge \neg R_2^1(x_2) \wedge \neg R_2^1(x_3) \wedge (\forall x_4)(\neg R_2^1(x_4) \supset (x_4 = x_1 \vee x_4 = x_2 \vee x_4 = x_3)\}$$

$$\text{pa}(6)\quad (\forall x_1)(\forall x_2)(R_2^1(x_1) \wedge R_2^1(x_2) \supset (f_{11}^1(x_1) = x_2 \supset x_1 R_1^2\, c_1)).$$

For the remainder of this section the notation $T(i_1, \ldots, i_k)$ will denote the subset of proper axioms consisting of $\{pa(i_1), \ldots, pa(i_k)\}$.

The notion of "intended interpretation" is made precise by defining a class of structures $\mathcal{M}_0$ whose members contain the particular functions and relations used in the analysis of this section. Let $\mathcal{M}_0$ be the collection of M's such that

$$M = \langle A \cup \mathbb{R} ;\ \geq, \mathbb{R}_2 ;\ P_A, p, \bar{p}, P_A^*, p^*, \bar{p}^*, +, -, \cdot, \div, \sqrt{\ } ;\ 0, 1 \rangle.$$

A is any three-element set, $\mathbb{R}_2$ is the membership relation on $\mathbb{R}$, P_A (and P_A^*) maps A into $\mathbb{R}$, p (and p^*) maps A x A into $\mathbb{R}$, $\overline{p}$ (and $\overline{p}^*$) maps A x A x A into $\mathbb{R}$, and the remaining functions and constants are defined on $\mathbb{R}$. That is, for any interpretation φ of sentences in L into a structure in $\mathcal{M}_0$, $\varphi(R_1^2) = \geq$, $\varphi(R_2^1) = \mathbb{R}_2$, $\varphi(f_7^2) = +$, $\varphi(f_{11}^1) = \sqrt{\ }$, $\varphi(c_1) = 0$, etc. Thus two structures M and M' in $\mathcal{M}_0$ differ only in the six "p" functions and in the choice of set A.

Within the class $\mathcal{M}_0$ there are structures which satisfy some of the various subsets of T(1,..., 6). $\mathcal{M}_0$ has been defined such that each M in $\mathcal{M}_0$ satisfies T(4, 5, 6). The structure defined in the barrel example, $M_b = \langle \{U, V, W\} \cup \mathbb{R}; \geq, \mathbb{R}_2, q_1, q_2, q_3, +, \ldots \rangle$, satisfies T(2, 4, 5, 6), but does not satisfy T(1, 2, 4, 5, 6). Theorem 2 states that any structure in $\mathcal{M}_0$ satisfying T(1,..., 6) must be one in which $P_A(x) = 1/3$ for all arguments x, $p(x, y) = 1/2$ for all x,y, and $\overline{p}(x, y, z) = 1/6$ for all x, y, z. Hence any two such structures M and M', since they differ only in having different sets A, A', are isomorphic. Thus the axiom set T(1,..., 6) has essentially one model in $\mathcal{M}_0$. Restated in this way Theorem 2 does not seem anomalous or contrary to experience. Similarly, Theorem 4 characterizes the subclass of $\mathcal{M}_0$ whose elements are models of T(1, 2, 4, 5, 6), and this subclass is also very small relative to $\mathcal{M}_0$.

Conclusion

At the end of the previous section, the class of models for the axiom set T(1, 2, 4, 5, 6) was described as being small compared to $\mathcal{M}_0$, even though both are countably infinite classes. In order to make sense out of that statement consider the inequality (17). It implies that for two specific values such as $\bar{p}(a, b, c)$ and $P_A(c)$, if their sum is greater than 1/2 by an amount α, then there is a subinterval or "hole" in the interval (0, 1) which must not contain either $\bar{p}(a, b, c)$ or $P_A(c)$. In fact, the size of that hole is given precisely as $\sqrt{2\alpha}$. However since the value of α depends upon the values of $\bar{p}$ and P_A, there is no fixed α which applies to all structures in $\mathcal{M}_0$, and thus (17) only gives a rough idea of the restrictions imposed by the axioms.

A better, but still intuitive, approximation on the relative size of the class of models can be derived from theorem 4. It states that only two probability values(which satisfy (17), e.g., $\bar{p}(a, b, c)$ and $P_A(c)$) need be known inorder to compute all the remaining values in a model. For an arbitrary structure in $\mathcal{M}_0$, however, five values for $\bar{p}$ would be needed in order to compute the sixth value using ca(2). In a very loose way, the class of models could be thought of as less than two-fifths the size of $\mathcal{M}_0$. Thus it would seem that these axioms put "more" constraints on the way choice probabilities may be related than has previously been thought to be the case.

As pointed out in the first section, the scarcity of models for an axiom set usually has led to attempts to identify "troublesome" axioms and remove them from the theory under consideration. However, that approach sometimes causes the loss of useful theorems (theorem 1, in our example). The point here is not whether it is desirable to have few or many models for a theory, but rather to offer an alternative to the axiom-elimination approach. In model-theoretic terms, that alternative is to enlarge or change the class of structures in which models may be sought.

Consider just the axioms pa(1) and pa(2) for a moment. The non-logical functions involved are just P_A, p, $\overline{p}$, +, and $\cdot$. The p functions must map into a set containing elements like 0 and 1 and a relation $\geq$ which has some order properties, in order to satisfy the calculus axioms, but otherwise there is no a priori restriction on how we might interpret those functions. A very large (with respect to $\mathcal{M}_0$) class $\mathcal{M}$ of structures could be defined as the class of M such that

$$M = \langle A \cup D; \geq, D_1; P_A, p, \overline{p}, +, \cdot; 0, 1\rangle$$

where A and D are two disjoint sets, D contains at least two distinct elements which we call 0, 1, P_A maps A into D; p maps A x A into D; $\overline{p}$ maps A x A x A into D; + and $\cdot$ are any two binary operations on D; $\geq$ is a binary relation in D; and D_1 indicates membership in D.

An example of such a structure is one in which D is the set of complex numbers, and the values of the function $\overline{p}$ (again denoted by $a_1, \ldots, c_2$) are

$$a_1 = \frac{3}{8} \qquad b_1 = \frac{-1 - 16\sqrt{2}i}{152} \qquad c_1 = \frac{3 - 6\sqrt{2}i}{40}$$

$$a_2 = \frac{1 + 2\sqrt{2}i}{8} \qquad b_2 = \frac{10 - 11\sqrt{2}i}{76} \qquad c_2 = \frac{12 + 6\sqrt{2}i}{40}$$

For these values, ${}^{\alpha}a_1 = 1/4$, which implies that a_1 is in one of the "holes" described above. Thus theorems 3 and 4 do not characterize the models of T(1, 2) in the class $\mathcal{M}$, and the class of models for T(1, 2) within $\mathcal{M}$ properly contains the corresponding class of models within $\mathcal{M}_0$. Finally we emphasize that it is questions such as these concerning the relationships between classes of set-theoretic structures to which the methods of mathematical model theory may be applied.

Footnotes

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